JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 24, No. 1, January–March 2010

# Technical Notes

## Unified Wien's Displacement Law in Terms of Logarithmic Frequency or Wavelength Scale

Z. M. Zhang* and X. J. Wang†
*Georgia Institute of Technology, Atlanta, Georgia 30332*

DOI: 10.2514/1.45992

### Nomenclature

| | | |
|---|---|---|
| $c$ | = | speed of light in vacuum, $2.998 \times 10^8$ m/s |
| $e$ | = | emissive power, W/m$^2$ |
| $h$ | = | Planck's constant, $6.626 \times 10^{-34}$ J · s |
| $k_B$ | = | Boltzmann's constant, $1.381 \times 10^{-23}$ J/K |
| $T$ | = | temperature, K |
| $\eta$ | = | logarithmic frequency |
| $\lambda$ | = | wavelength, m |
| $\nu$ | = | frequency, Hz |
| $\xi$ | = | logarithmic wavelength |

*Subscripts*

| | | |
|---|---|---|
| $b$ | = | blackbody |
| $n$ | = | power of $\nu$ or $\lambda^{-1}$ in the distribution functions |
| ref | = | reference frequency or wavelength |
| $\eta$ | = | logarithmic frequency |
| $\lambda$ | = | wavelength |
| $\nu$ | = | frequency |
| $\xi$ | = | logarithmic wavelength |

### I. Introduction

IT IS well known that Wien's displacement law takes a different form when Planck's law is expressed in terms of frequency than that in terms of wavelength (in vacuum) [1,2]. This can be explained by the different functional relationships because the wavelength is inversely proportional to the frequency of electromagnetic radiation. By introducing a logarithmic frequency or wavelength scale, a unified Wien's displacement law is obtained regardless of whether the frequency or wavelength is used as the independent variable. The new characteristic wavelength is approximately 26.6% longer than the conventional Wien's displacement law in terms of wavelength.

### II. Existing Theory

Planck's law can be derived from Bose–Einstein statistics of a phonon gas in a three-dimensional isothermal enclosure (blackbody cavity) with two polarizations. The result can be expressed as [1–3]

$$e_{b,\nu}(\nu,T) = \frac{2\pi h\nu^3}{c^2[\exp(\frac{h\nu}{k_B T}) - 1]} \tag{1}$$

Here, $e_{b,\nu}(\nu,T)$ is the emissive power, $T$ is the temperature of the cavity measured by its wall temperature, $\nu$ is the frequency, $c$ is the speed of light in vacuum, $h$ is Planck's constant, and $k_B$ is Boltzmann's constant. In terms of wavelength, Planck's law can be expressed as

$$e_{b,\lambda}(\lambda,T) = \frac{2\pi hc^2}{\lambda^5[\exp(\frac{hc}{k_B\lambda T}) - 1]} \tag{2}$$

Equation (2) can be derived from Eq. (1) using

$$e_{b,\nu}\,\mathrm{d}\nu = -e_{b,\lambda}\,\mathrm{d}\lambda \tag{3}$$

because $\nu = c\lambda^{-1}$, and $\mathrm{d}\nu = -c\lambda^{-2}\,\mathrm{d}\lambda$.

By setting the derivative of Eq. (1) with respect to frequency to zero, one obtains

$$x_3 = \frac{h\nu_3}{k_B T} = \frac{hc}{k_B\lambda_3 T} = 2.8215$$

or

$$\lambda_3 T = 5099.3\ \mu\mathrm{m} \cdot \mathrm{K} \tag{4}$$

Similarly, by differentiating Eq. (2) with respect to wavelength and setting it to zero, one obtains

$$x_5 = \frac{hc}{k_B\lambda_5 T} = 4.9651$$

or

$$\lambda_5 T = 2897.7\ \mu\mathrm{m} \cdot \mathrm{K} \tag{5}$$

Equation (5) is known as Wien's displacement law in most heat transfer texts [1–4]. In Eq. (4) or Eq. (5), subscript 3 or 5 signifies the power of $\nu$ or $\lambda^{-1}$ in the distribution functions given, respectively, by Eq. (1) or Eq. (2). If a compromise were made between Eqs. (1) and (2) by using a fourth-power function, one would end up with the following relation:

$$\lambda_4 T = 3669.7\ \mu\mathrm{m} \cdot \mathrm{K} \tag{6}$$

This has been called a wavelength-frequency-neutral peak [5]. Notice that in all of the previous equations

$$x_n = \frac{hc}{k_B\lambda_n T}$$

is the root of the transcendental equation

$$(n - x_n)\exp(x_n) - n = 0 \tag{7}$$

where $n$ is a positive integer. Equation (7) was obtained by setting the derivative of the following function with respect to $x$ to zero:

$$f_n(x) = \frac{x^n}{\exp(x) - 1} \tag{8}$$

Interestingly, if two-dimensional space is considered, the density of states (DOS) will be reduced from $D_{3D}(\nu) = 8\pi\nu^2/c^3$ to $D_{2D}(\nu) = 4\pi\nu/c^2$, as is often done when studying the specific heat of a two-dimensional solid [3,6]. For one-dimensional space, the DOS becomes independent of frequency [3]. This is to say that in the

Received 15 June 2009; revision received 9 September 2009; accepted for publication 10 September 2009. 

*Professor, George W. Woodruff School of Mechanical Engineering; zhuomin.zhang@me.gatech.edu. Associate Fellow AIAA (Corresponding Author).

†Graduate Research Assistant, George W. Woodruff School of Mechanical Engineering.

two-dimensional case, Planck's function is proportional to $\nu^2$ in the frequency domain but $\lambda^4$ in the wavelength domain. Furthermore, in the one-dimensional case, Planck's function will be proportional to $\nu$ in the frequency domain but $\lambda^{-3}$ in the wavelength domain. It is noted that there does not exist a finite solution for $x_1$ because $f_1(x)$ is a monotonically decreasing function of $x$. On the other hand, the peak does exist for the one-dimensional case in the wavelength domain, and the location is at $x_3$.

## III. Proposed Distribution Functions

The previous discrepancies can be resolved by using a logarithmic frequency or logarithmic wavelength as the variable. Let $\eta = \log_{10}\nu$, where $\nu$ is in Hz, and $\xi = \log_{10}\lambda$, where $\lambda$ is in nm. One can define $\eta = \log_{10}(\nu/\nu_{\text{ref}})$ with $\nu_{\text{ref}} = 1$ Hz and $\xi = \log_{10}(\lambda/\lambda_{\text{ref}})$ with $\lambda_{\text{ref}} = 1$ nm. Note that $\eta$ and $\xi$ are linearly related by

$$\xi = \log_{10}(c) - \eta = 17.4768 - \eta \tag{9}$$

where the unit of $c$ is adjusted to nm/s. From Eq. (1), it can be seen that

$$e_{b,\nu}(\nu, T)\,\mathrm{d}\nu = \frac{2\pi h\nu^3}{c^2[\exp(\frac{h\nu}{k_B T}) - 1]}\,\nu\,\ln(10)\,\mathrm{d}\eta \tag{10}$$

which can be expressed as

$$e_{b,\eta}(\eta, T) = \frac{2\pi b h\nu^4}{c^2[\exp(\frac{h\nu}{k_B T}) - 1]} \tag{11}$$

Here, $b = \ln(10)$ and $\nu = \nu_{\text{ref}}10^{\eta}$. The substitution of $\eta$ for $\nu$ was not made on the right side of the equation to keep the form simple. One may choose to use natural logarithm to define $\eta$ to remove $\ln(10)$ in the previous equation, but the value of $\eta = \log_{10}(\nu/\nu_{\text{ref}})$ can be easily related to the actual frequency when plotted. Similarly, Planck's distribution function in terms of $\xi$ is

$$e_{b,\xi}(\xi, T) = \frac{2\pi b h c^2}{\lambda^4[\exp(\frac{hc}{k_B\lambda T}) - 1]} \tag{12}$$

where $\lambda = \lambda_{\text{ref}}10^{\xi}$ is in nm. The peaks for both Eqs. (11) and (12) occur at the same frequency or wavelength location, which is prescribed by Eq. (7) with $n = 4$. In terms of wavelength, the peak is given by Eq. (6).

The previous derivation offers a justification for using Eq. (6) as the wavelength-frequency-neutral peak or the unified Wien's displacement law. The distribution function in terms of $\xi$ calculated from Eq. (12) is shown in Fig. 1 at various temperatures. Because $\xi$ is dimensionless, $e_{b,\xi}(\xi, T)$ is in W/m$^2$, which is the same as the total emissive power. Similar to the classical Planck distribution of spectral blackbody emissive power, the peak of the emissive power shifts to smaller $\xi$ (shorter wavelengths) at higher temperatures. The peak emission for a blackbody near room temperature (300 K) occurs at $\xi = 4.09$, which corresponds to 12.2 $\mu$m and is slightly shifted toward longer wavelength (by 26.6%) as compared with that predicted by the original Wien's formula given in Eq. (5). For solar radiation, which can be approximated as a blackbody at 5800 K, the peak emission occurs at $\xi = 2.80$ with a peak wavelength of 632.7 nm.

When Eq. (12) is normalized by $\sigma T^4$ and plotted against $\log_{10}(\lambda T/\mu\text{m}\cdot\text{K})$, all curves in Fig. 1 merge into one curve as shown in Fig. 2. It should be noted that for fixed temperature, the derivatives of $\log_{10}(\lambda)$ and $\log_{10}(\lambda T)$ with respect to wavelength are the same. The area under the normalized distribution function is one. Furthermore, the normalized distribution function can be integrated from 0 to a given $\lambda T$ to give the cumulative distribution function, which is also shown in Fig. 2.

The use of logarithmic frequency or wavelength scale has several other advantages:

1) The physical meaning of Eq. (1) is the emitted energy per unit frequency interval and that of Eq. (2) is the emitted energy per unit wavelength interval. In many practical applications, the bandwidth is specified in terms of the relative variation of the frequency $\nu^{-1}\Delta\nu$ or wavelength $\lambda^{-1}\Delta\lambda$. Equations (11) and (12) give the distribution functions in terms of the relative variation of frequency and wavelength, respectively, because $\mathrm{d}\eta$ is proportional to $\nu^{-1}\,\mathrm{d}\nu$ and $\mathrm{d}\xi$ is proportional to $\lambda^{-1}\,\mathrm{d}\lambda$.

2) As shown in Fig. 1, the abscissa gives the wavelength to the tenth power. The wavelength region for thermal radiation is generally from $\xi = 2$ to 6. When plotted on a linear scale according to $\xi$, the area under the curve in the given spectral region represents the blackbody emissive power within that band. This is not the case when Eq. (2) is plotted against the wavelength on a log scale, as is usually done in thermal radiation texts [1,2].

3) The fraction of energy emitted by a blackbody from $0 < \lambda < \lambda_5$ is 25.0%, suggesting that the peak in the conventional Wien's law divides the total energy 1:3 between the shorter and longer wavelength regions. This ratio is independent of which variable is used to express Planck's law. When plotted on a logarithmic scale, the peak wavelength $\lambda_4$ is given by Eq. (6). The energy is divided more evenly in the shorter ($\lambda < \lambda_4$) and longer ($\lambda > \lambda_4$) wavelength regions, with approximately 41.8% in the shorter wavelength region, as shown in Fig. 2, by the cumulative distribution function. Hence, the new peak divides the blackbody radiation more evenly between the shorter and longer wavelength regions.

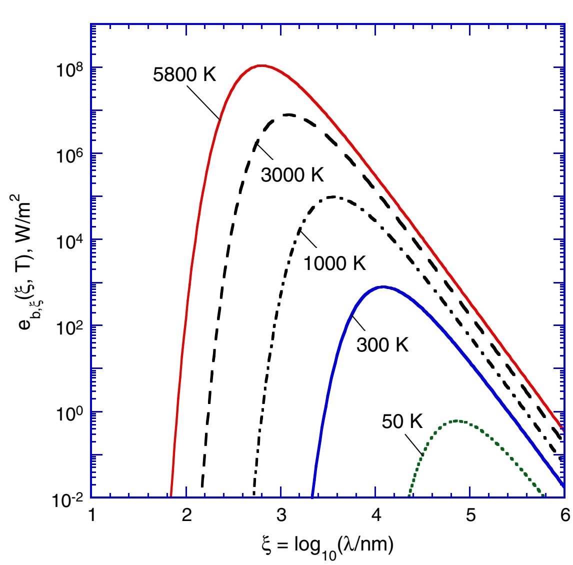


**Fig. 1 Emissive power in terms of the logarithmic wavelength scale $x$.**

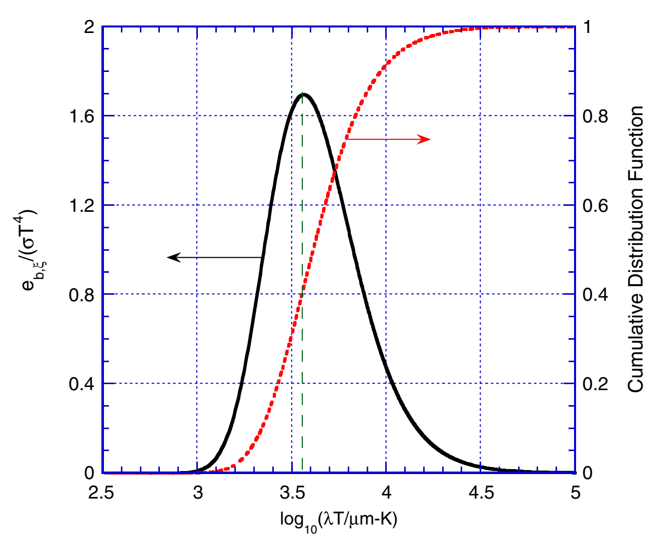


**Fig. 2 Normalized emissive power and the cumulative distribution function.**

4) In near-field radiation, sometimes it is necessary to evaluate the peak of the mean energy of a Planck oscillator, which is a function similar to $f_1(x)$ given in Eq. (8). Unfortunately, this function does not have a peak because it is a monotonically decreasing function. When the distribution function is converted to logarithmic frequency (or wavelength), it becomes $f_2(x)$, which has a peak at

$$\lambda_2 T = 9034.6\ \mu\mathrm{m}\cdot\mathrm{K} \tag{13}$$

The use of Eq. (13) allows the optimization of the dielectric function for nanoscale thermal radiation [7,8].

## IV. Conclusions

A logarithmic frequency or wavelength scale is proposed for the Planck blackbody distribution, whose peak location is the same in terms of both $\log_{10}(\nu)$ or $\log_{10}(\lambda)$. The consequence is a unified Wien's displacement law that is no more dependent on whether wavelength or frequency is chosen as the variable to express Planck's law. The results can be easily extended to two-dimensional and one-dimensional Planck distributions and may be useful in analyzing near-field radiative transfer.

## Acknowledgment

This work was supported by the National Science Foundation (CBET-0828701).

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